

Fluctuations in horizon-fluid lead to negative bulk viscosity

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Einstein equations projected on to a black hole horizon gives rise to Navier-Stokes equations. Horizon-fluids typically possess unusual features like negative bulk viscosity and it is not clear whether a statistical mechanical description exists for such fluids. In this work, we provide an explicit derivation of the Bulk viscosity of the horizon-fluid based on the theory of fluctuations *à la* Kubo. The main advantage of our approach is that our analysis remains for the most part independent of the details of the underlying microscopic theory and hence the conclusions reached here are model independent. We show that the coefficient of bulk viscosity for the horizon-fluid matches exactly with the value found from the equations of motion for the horizon-fluid.

I. INTRODUCTION

In 1970's, it was found that the dynamics of Black-holes is formally analogous to thermodynamics and is now widely viewed as the laws of black-hole thermodynamics [1, 2]. This offered support for a proposal made earlier by Sakharov that Gravity may be an emergent phenomenon [3–6]. Around the same time, it was shown that the horizon of a black hole behaves like a viscous fluid i. e., horizon obeys Navier-Stokes equation [7, 8]. Over the last few years, a growing body of evidence suggests that gravitational dynamics near the black-hole horizon, is analogous to the dynamics of a viscous fluid[9–11].

The current status of the fluid-gravity correspondence appears somewhat similar to the laws of black-hole mechanics in early 70's. There seems to be a lot of similarities between the equations of Gravity on the horizon and fluid mechanics. However, it is unclear whether the mathematical similarities of the two sets of equations describing two completely different systems suggest something more physical. In particular, the question that needs to be answered is whether the fluid-gravity correspondence can provide a statistical-mechanical description of the degrees of freedom (DOF) that leads to black-hole entropy for a general black-hole spacetime.

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Such a program runs into difficulties primarily because the horizon-fluid does not behave like a normal fluid. In particular, it has a negative bulk viscosity (ζ) for asymptotically flat space-times like Schwarzschild, Kerr [7]. For asymptotically AdS black-holes, for which the horizon-fluid has positive bulk viscosity, there seems to exist such a description via a CFT on the boundary [9, 12]. However, it is not possible to provide a statistical mechanical description for the horizon-fluid corresponding to asymptotically flat space-times. For asymptotically flat space-times, we do not yet have any powerful physical principle that can guide us from a microscopic description of Gravity to a fluid description. To be more specific, there does not exist a microscopic theory dual (that exists for asymptotically AdS space-times) to the theory of Gravity for such space-times¹.

In view of this, we shall adopt a different approach. We assume that the horizon-fluid undergoes statistical fluctuations and construct a statistical mechanical description of the fluid based on the theory of fluctuations [15–17]. We show that this approach provides mathematical tools to address some of the open problems in Fluid-Gravity correspondence. The dynamical phenomena that can be described by the theory of fluctuations mainly constitute a class of processes where the system is typically slightly away from equilibrium and are broadly known as Transport phenomena. In particular, the diffusion of heat, electrical conductivity, and the shear and the bulk viscosity of the fluid fall under this category.

In this approach, we take the view that the horizon-fluid possesses some kind of physical reality beyond the formal similarity. One advantage of this strategy is that our analysis would remain for the most part independent of the details of the underlying microscopic theory and hence the conclusions reached here would be model independent. It also follows that in the absence of any microscopic theory, our approach here can only be phenomenological. The whole analysis is performed within the Mean Field Theory model of the horizon-fluid [18, 19]. However the main role of Mean Field Theory here is to motivate a choice of a macroscopic variable in terms of which, one can write down the Thermodynamic Potential for the fluid system. In some other model, some other macroscopic variable might have to be chosen. As will be clear, this does not change the physical analysis.

In what follows, we shall focus on the transport phenomena related to the bulk viscosity of the fluid. This is due to two reasons. First, for bulk viscosity, we need to consider only a homogeneous fluid system for the most part. This allows us to ignore the effect of shear and simplifies the analysis to a great extent. Second and more importantly, the Coefficient of bulk viscosity is negative for the

¹ See [14] for an interesting attempt in this direction.

horizon-fluid and the reason for this is not well-understood at present; it constitutes a challenge for any statistical mechanical description of the horizon-fluid to compute this quantity. Here we explicitly compute it using the theory of fluctuations and taking into account the teleological nature of the Black Hole event horizon [8, 20].

The paper is organized as follows. In section II, we provide a broad outline of our approach to construct a theory of fluctuations for the horizon-fluid. In section III, we discuss the Mean Field Theory model of the horizon-fluid and briefly review the earlier results that are essential to compute the transport coefficients. Section IV is the main part of the work where we explicitly compute the Coefficient of bulk viscosity for the horizon-fluid. Finally, in section V, we discuss the implication of our results.

II. THEORY OF FLUCTUATIONS FOR THE HORIZON-FLUID: A BROAD OUTLINE

In this section, we give a broad outline of the approach we follow to study the fluctuations and transport phenomena for the horizon-fluid. The four main ingredients that go in are as follows:

1. The macroscopic properties of the horizon-fluid must satisfy the black-hole constraints. For a Schwarzschild black-hole, the fluid must satisfy relations between total energy(E), pressure(P), temperature(T), horizon area(A) and the number of d.o.f.(N).
2. The macroscopic properties of the horizon-fluid are specified by the collective properties of its microscopic constituents. The microscopic constituents result in statistical fluctuations of the macroscopic quantities. Such fluctuations will influence the evolution of a system from a non-equilibrium state to a state in equilibrium.
3. Any dynamical process that the event horizon of a black-hole undergoes corresponds to the horizon-fluid system moving from an initial non-equilibrium state towards an equilibrium state. However, so long as the evolution of the event horizon follows from a classical theory of Gravity, which is what we shall look at here, the system is not far from equilibrium in the corresponding fluid picture.

The above assumption imposes a restriction on the kind of dynamical processes we consider on the Gravity side. In the fluid side, we are looking at processes that increase the entropy of the system. However, on the Gravity side, it is the area of the event horizon that is proportional to the entropy[1, 2, 21] and even in a classical theory, it does not always increase. We do not take into account here the semiclassical effects like the decrease of black-hole area due to

Hawking radiation [2, 21] as those would be quite small for large black-holes. In the classical theory, in order for the area of the event horizon to increase, some additional conditions have to be imposed[8], like the condition that the black-hole is strongly asymptotically predictable [21, 22].

4. The statistical fluctuations of the system are assumed to obey the Principle of Equipartition and Maxwell-Boltzmann Statistics.

For the fluid description, the relevant physical quantities vary slowly in space and time, a situation we will consider here. The fluid-dynamical equations contain parameters, whose values cannot be determined from fluid mechanics. These arise due to the changes in the macroscopic variables of the system as a response to the external influences that are not large. These parameters fall into two classes: (i) Thermodynamic derivatives which relate the changes in local Thermodynamic variables and (ii) transport coefficients like viscosity and thermal conductivity that relate the fluxes of thermodynamic quantities to the gradients of the local variables. These parameters can only be obtained from a more fundamental theory². The theory of fluctuations that we construct here for the horizon-fluid works at this level.

Statistical fluctuations of normal fluids is well-understood [23]. The horizon-fluid, on the other hand, is an unusual fluid. Unlike normal fluids, it is a one parameter system, whose energy, pressure, temperature, volume(area in this case) are not independent of each other. It also has negative bulk viscosity. Here we explicitly show that it is possible to construct a theory of the Fluctuations in the Horizon Fluid in analogy with normal fluids. As will be seen, our approach is quite general and does not depend on the particular Mean Field Theory model that we use to compute the Coefficient of bulk viscosity (ζ).

The basic strategy to construct the theory and compute transport coefficients is as follows:

1. The horizon-fluid for a Schwarzschild black-hole is a one parameter system[18, 24]. We identify a relevant macroscopic variable and write down the Thermodynamic Potential in terms of it. The equilibrium state corresponds to a minimum of the Thermodynamic Potential.
2. The fluctuations about the equilibrium position of the system are about this minimum value. This implies that the leading order variation in the potential would be of second order in that variable.

² See Ref. [15] for a good discussion on this topic. Our treatment of this issue is very much in the same spirit.

3. We write down a Langevin equation that governs the transport processes to be considered.
4. We make the assumption that the random fluctuations that the system undergoes are of a much shorter time-scale than the time period over which the system goes over from one state to another due to external influence. These latter processes can also be viewed as fluctuations that the system undergoes; only they are of a much longer time-scale. This makes the probability distributions for the random short-time fluctuations and the long time ones independent of each other. It is possible to compute the correlation functions that contain all the information about the system from the assumptions that the random fluctuations obey the principle of Equipartition and both the probability distributions are given by Maxwell-Boltzmann. Another necessary input is the black-hole constraints, which reflects the fact that Horizon fluid is actually a one parameter system.
5. The Transport coefficients for the horizon-fluid can then be computed using a method given by Kubo[16] either in real space or in frequency space.

III. MEAN FIELD THEORY AND FLUCTUATIONS ABOUT THE EQUILIBRIUM

In this section, we provide key results of Refs. [19] that forms the basis for computing the transport coefficients. In particular, we shall sketch the Mean Field Theory for the horizon-fluid for a Schwarzschild black-hole. After this, we shall set up a theory of the statistical fluctuations within a Mean Field Theory setup. In the first subsection, we list all the useful relations of black-hole thermodynamics.

A. Constraints from black-hole Physics

The Einstein equations projected on the event horizon of a Schwarzschild black hole can be described by a $(2+1)$ -dimensional fluid that resides on the event horizon of the black hole [7]. In particular, in Damour's approach, the Navier-Stokes equation³ governing the horizon-fluid dynamics is given by:

$$\frac{D\Pi_A}{dt} = -\frac{\partial}{\partial x^A}\left(\frac{\kappa}{8\pi}\right) + 2\frac{1}{16\pi}\sigma_{A|B}^B - \frac{1}{16\pi}\frac{\partial\theta}{\partial x^A} - l^a T_{aA}, \quad (1)$$

³ Actually the equation found by Damour is not exactly same as the Navier-Stokes equation but very similar to it.

where, σ is the shear tensor for the null congruence on the horizon, \mathbf{l} is the vector normal to the horizon and x^A denotes the coordinates on the null hypersurface perpendicular to the null generators of the horizon. The vertical bar denotes the covariant derivative with respect to the induced metric on the null hypersurface. $\mathbf{\Pi}$ can be thought of as the momentum density of the horizon-fluid. The coefficient of bulk (ζ) and Shear viscosity (η), respectively, are given by

$$\zeta = -\frac{1}{16\pi}; \eta = \frac{1}{16\pi}. \quad (2)$$

The volume of the fluid is the area of the horizon (A). The temperature of the horizon (T) is the temperature of the fluid and the total energy of the fluid is given by the Komar mass of the black hole [24]. The equation characterizing the fluid is given by [24],

$$P = \frac{T}{4} = \frac{E}{2A}. \quad (3)$$

The parameter space of the Schwarzschild black-hole is one dimensional. So E , T , A are not independent, but must obey the two constraints: $E = E(A)$ and $E = E(T)$. The constraint equation between E and A is given by [24],

$$A = 16\pi E^2. \quad (4)$$

The constraint equation relating the black hole mass and the Hawking temperature is

$$E = \frac{1}{8\pi T}. \quad (5)$$

B. The Mean Field Theory model for the horizon-fluid

We start by assuming that the Horizon-fluid forms a condensate at a critical temperature. The justification comes from two lines of arguments. First, the evidence provided by Carlip[25] that the black-hole horizon has some properties that exhibit universality. This indicates that the physics near the horizon is that of a system near a critical point. Second, recently, Skakala and Shankaranarayanan[24] modelled the fluid as a Bose gas with N particles and found that all the particles stayed in the ground state for large horizon radius. This suggests that horizon-fluid forms a BEC at some critical temperature T_c . The conditions underlying this are [18]:

1. There is a temperature T_c (critical temperature), at which, all the N fluid degrees of freedom on the horizon form a condensate.
2. The system remains close to the critical point.

An immediate consequence of these assumptions is the deduction of the relation between N and A . Since the system forms a condensate at T_c , nearly all the microscopic d.o.f.s would be in the ground state near the critical point. As there is only one scale in the problem, the total energy of the system can be expressed in the form,

$$E = N\alpha T, \quad (6)$$

where, α is a constant. Using the other black-hole constraints, one can write a relation between N and A ,

$$N = \frac{A}{2\alpha}, \quad (7)$$

One can describe this critical system, a homogeneous fluid, using Mean Field Theory. Taking the order parameter to be

$$\eta = \sqrt{kN}. \quad (8)$$

The thermodynamic Potential Φ is

$$\Phi = \Phi_0 + a(P)(T - T_c)\eta^2 + B(P)\eta^4. \quad (9)$$

where k is a constant, coefficients a and B are determined by the relation $P = -TA$. One can determine the entropy of the fluid in $\eta \neq 0$ phase and show that it leads to the Bekenstein-Hawking entropy, $S = \frac{A}{4}$ [18]. The Horizon-fluid is near a critical point and when the system goes over to the ordered phase, its entropy is the same as the black-hole entropy. This formalism can be extended to include black-holes in AdS background[18].

C. Fluctuations about the Equilibrium Position

What has been described so far, corresponds only to static situations in the gravity theory. The dynamics of the black-hole event horizon can be described by the equations of motion of a fluid. The basic idea that underlies our Statistical Mechanical description of the Horizon Fluid is that the dynamical evolution of the black-hole event horizon in the fluid picture corresponds to the fluid system moving from a state slightly away from the equilibrium towards a state in equilibrium [19].

When one does not look at the dynamics of the evolution and only considers the initial and the final states, then the change in the energy and the entropy of the fluid in the process are seen to be related via the First Law of Thermodynamics. This corresponds to the Physical Process First

Law of black-hole Thermodynamics [See Appendix A]. However, if one considers how the dynamical evolution of the fluid takes place, then such a process can be described by a Langevin equation. It can then be shown that this equation is similar to the energy conservation equation of the Horizon Fluid or the Raychaudhuri equation for the null congruences on the horizon[See Appendix A].

Let us now determine how the fluctuations from the equilibrium position can be expressed within the Mean field Theory. The one parameter Schwarzschild Black hole system corresponds to the fluid system at equilibrium. Fluctuations, assumed here to be isothermal, do not obey all the constraint equations. As this is a one parameter system, the fluctuations can be characterised by the deviations of the order parameter η around its mean value at equilibrium. This implies that not all the constraint equations are satisfied when the system is away from equilibrium. In the Mean field Theory, the equilibrium state corresponds to any one of the minima of the double well for the Thermodynamic Potential Φ of the horizon-fluid system. This state is described by the value of the order parameter η ,

$$\eta_{min}^2 = \frac{a(T_c - T)}{2B}. \quad (10)$$

Let, $\delta\eta(\delta N)$ denote the change in the value of the order parameter(number) due to fluctuations, then the change in the potential is

$$\delta\Phi = \frac{1}{4}\alpha(T - T_c)\frac{\delta N^2}{N_0}. \quad (11)$$

This relation will be useful when we compute the coefficient of Bulk Viscosity.

IV. BULK VISCOSITY OF THE HORIZON-FLUID

In this section, using the theory of fluctuations [15, 16], we compute the coefficient of bulk viscosity (ζ) of the horizon-fluid.

The transport coefficient for a particular process can be determined by identifying the relevant current and computing the autocorrelation function of the current [15, 16, 23, 26, 27]. In the case of the horizon-fluid, we need to identify the current corresponding to the change in the fluid volume (area in this case). Unlike normal fluids, the horizon-fluid is a one parameter system. Hence, it is not completely straightforward to generalise the formula of the ordinary fluid to the case of the horizon-fluid.

We proceed in the following manner: (i) Obtain the formula for ζ for a normal fluid and relate the corresponding current to an entropic force acting on the system. (ii) Obtain ζ in terms of

an autocorrelation function of the entropic force for a normal fluid. (iii) Extend the formula to compute ζ for the one parameter horizon-fluid system for anti-causal transport processes.

A. ζ for a normal fluid in Fluctuation Theory

For a 3-dimensional Newtonian fluid, Kubo's method [16] can be used to calculate ζ . If the particles obey the Principle of Equipartition and follow Maxwell-Boltzmann statistics, then ζ is given by [15],

$$\zeta = \left(\frac{1}{9}\right) \frac{1}{VKT} \int_0^\infty dt \sum_a \sum_b \langle J^{aa}(0) J^{bb}(t) \rangle, \quad (12)$$

where,

$$J^{ab} = \delta_{ab} \delta(PV) = V \delta P \delta_{ab}. \quad (13)$$

δ denotes the variation in any quantity [23, 26] and the factor $\frac{1}{9}$ comes from the number of dimensions (3-dimensions). The range of integration is between $[0, \infty)$ as only causal processes are being considered. It is to be noted that the range of integration will be different for a horizon-fluid.

Substituting (13) in (12), we get,

$$\zeta = \left(\frac{1}{9}\right) \frac{V}{KT} \sum_a \sum_b \delta_{aa} \delta_{bb} \int_0^\infty dt \langle \delta P(0) \delta P(t) \rangle = \frac{V}{KT} \int_0^\infty dt \langle \delta P(0) \delta P(t) \rangle. \quad (14)$$

Keeping in mind that our primary concern for this work is the horizon-fluid — which is highly constrained — the change in the pressure of the fluid can then be related to the entropic force acting on the system. We show this [28] by assuming an idealised situation, where the system is isotropic and resides inside a cube. Let $\delta V_{Strain} = \delta V_{Compress}/V$. Let us assume that the energy dissipated in this process is given by δE_d and

$$\delta E_d = -F_{Th} \delta V_{Strain}, \quad (15)$$

where, F_{Th} is the entropic force. If we assume that the work done on the system is totally lost i. e. it gets converted to heat, then due to the isotropy of the system, one can write,

$$\delta E_d = -3\delta P \delta V_{Compress} = (3\delta PV) \delta V_{Strain}. \quad (16)$$

Comparing (15) with (16), we identify,

$$F_{Th} = 3\delta PV. \quad (17)$$

Recalling the expression for the current (13), we identify entropic force F_{Th} as

$$F_{Th}^{ab} = J^{ab} = \delta P V \delta_{ab}. \quad (18)$$

Substituting the above expression in (14), we get,

$$\zeta = \frac{1}{VKT} \int_0^\infty dt \langle F_{Th}(0) F_{Th}(t) \rangle. \quad (19)$$

This expression is also valid for the horizon-fluid with modified limits of integration.

B. ζ for the horizon-fluid in Fluctuation Theory

Evaluation of ζ for the horizon-fluid will be done in two steps: First, we shall write down the entropic force acting on the horizon-fluid, when the system is not in equilibrium. Then we have to evaluate an autocorrelation function for the entropic force, which requires a crucial input about the horizon properties.

1. Entropic force for the horizon-fluid

As discussed earlier, we associate the deviation from the equilibrium to the fluctuation around one of the minima of the thermodynamic potential, Φ of the horizon-fluid. The change in the potential ($\delta\Phi$) is given by (11). The energy lost in this process is the heat (δQ) generated during the process. δQ can be determined from the First Law of Thermodynamics, $\delta Q = T\delta S$. From (11), the change in entropy in this process comes out to be

$$\delta S = -\frac{1}{8} \frac{\delta A^2}{A}. \quad (20)$$

The negative sign of the change in entropy denotes that the system is out of equilibrium. Since $\delta E_d = \delta Q$, we have

$$\delta E_d = \frac{T}{8} \frac{\delta A^2}{A} = \frac{T}{8} \left(\frac{\delta A}{A} \right)^2 A. \quad (21)$$

From (21), we get,

$$d(\delta E_d) = \frac{T}{4} \frac{\delta A}{A} d(\delta A) \quad (22)$$

The above expression is of the form

$$d(\delta E_d) = F_{Th} d(\delta A_{Strain}). \quad (23)$$

Comparing (22) and (23), gives

$$F_{Th} = P \delta A. \quad (24)$$

2. The Teleological nature of event horizon and Anti-causal Transport Process in the horizon-fluid

As mentioned earlier, only causal response is seen in normal-fluids in the presence of external influence. However, the same is not true for the horizon fluid. To understand why this happens, let us first look at the evolution of the black-hole event horizon. For the event horizon, the response to any external influence is anti-causal. In particular, if matter- energy falls through the event horizon, then the area of the event horizon increases till the matter-energy passes through the horizon. This is not unphysical as the event horizon of a black hole is defined globally in the presence of the future lightlike infinity [8]

Due to this unusual property of the horizon, the horizon-fluid also exhibits anti-causal response, i.e. the response of the horizon takes place before the external influence occurs[8]. Since from the fluid point of view, the system is initially out of equilibrium and slowly moves towards the final state in equilibrium, it follows that the external influence brings the system to equilibrium, so that there is no further evolution of the system from the state it is in. This is referred to as the teleological nature of horizon [8].

For a class of systems, it has been shown in the literature that if the system exhibits anti-causal transport process, then the anti causal transport coefficients have an opposite sign to their causal counterparts[20]. For normal fluids, external influence drives the system out of equilibrium. For the horizon-fluid, it is the reverse; the system moves towards equilibrium in anticipation of the external influence like infusion of energy into the fluid. This is the anti-causal response of the horizon-fluid.

The anti-causal response can be incorporated by defining $\delta A_a(t)$ as $\delta A_a(t) = \delta A(t)\theta(-t)$ and the anti-causal entropic force ($F_{Th}^{(a)}$) is then given by,

$$F_{Th}^{(a)}(t) = PA_0\delta A_a(t) = PA_0\delta A(t)\theta(-t). \quad (25)$$

One can also arrive at the same relation using Green's function to calculate ζ . When determining ζ , we have to determine the autocorrelation function of $F_{Th}^{(a)}(t)$.

3. Computation of ζ using Kubo's method

In the rest of this section, we compute ζ in the real space. In Appendix B, we compute the same in the Frequency space. It is important to note that both the computations lead to the same value of ζ . ζ is given by,

$$\zeta = \frac{1}{AKT} \int_{-\infty}^{\infty} dt \langle F_{Th}^{(a)}(t) F_{Th}^{(a)}(0) \rangle. \quad (26)$$

Substituting for $F_{Th}^{(a)}$ from (25), we get

$$\zeta = \frac{P^2}{AKT} \int_{-\infty}^{\infty} dt \langle \delta A(t) \delta A(0) \rangle \theta(-t). \quad (27)$$

Using (3) and (7), (27) can be written as

$$\zeta = \frac{\alpha^2 T}{4AK} \int_{-\infty}^{\infty} dt \langle \delta N(t) \delta N(0) \rangle. \quad (28)$$

To proceed further, one needs to know the functional form of $\delta N(t)$. Since our goal here is to provide an analytical expression for the transport coefficients of the fluid, which corresponds to the long wavelength limit of the microscopic theory [15, 23, 27], the long wavelength (small frequency) limit of the above expression leads to

$$\zeta = \lim_{\epsilon \rightarrow 0} \text{Im} \left[\frac{\alpha^2 T}{4AK} \langle \delta N^2(0) \rangle \int_{-\infty}^{\infty} dt \exp i(\omega - i\epsilon)t \theta(-t) \right]. \quad (29)$$

Hence,

$$\zeta = -\frac{\alpha^2 T}{4AK} \frac{\langle \delta N^2(0) \rangle}{\omega}. \quad (30)$$

To obtain ζ , we need to determine $\langle \delta N^2(0) \rangle$ and ω . Note that ω corresponds to the lowest possible frequency of the system, which for the horizon-fluid is related to the horizon cross-section. For Schwarzschild, we have

$$r_h = 2E \quad (31)$$

Using (5), we can write this as

$$r_h = \frac{1}{4\pi T}. \quad (32)$$

The circumference, s of the black-hole cross-section is given by $s = 2\pi r_h = \frac{1}{2T}$. Then the wavenumber k of the wave modes is

$$k = \frac{2\pi}{\lambda} = 4\pi T. \quad (33)$$

Hence,

$$\omega = k = 4\pi T. \quad (34)$$

To determine $\langle \delta N^2(0) \rangle$, we need to assume a probability distribution for $\delta N(0)$. Assuming Maxwell-Boltzmann Statistics, the probability distribution is given by, $N_c \exp(\delta S)$, where, N_c is

a normalisation constant[29]. Using (20), it can be written as $N_c \exp -\frac{1}{8} \frac{\delta A^2}{A_0}$. Switching to the variable δN , this can be written as $N_c \exp \left(-\frac{\alpha^2}{2} \delta N^2 A_0 \right)$. This implies that[29]

$$\delta N^2(0) = \frac{A_0}{\alpha^2}. \quad (35)$$

From equations (30), (34) and (35); we get,

$$\zeta = -\frac{1}{16\pi}, \quad (36)$$

exactly the value read off from the Navier-Stokes equation for the horizon-fluid. It is to be noted here that the Principle of Equipartition has been implicitly used here in writing down the formula for ζ given by (26).

C. Positive Shear Viscosity Coefficient from the Theory of Fluctuations

So far, we have shown that the anti-causal or teleological nature of the change in the area of the Horizon-fluid is responsible for the negative coefficient of Bulk Viscosity of the Horizon-fluid. Our analysis can not be extended to obtain the coefficient of Shear Viscosity. To get a better understanding, let us look at the evolution equations for the volume expansion coefficient and the shear of a null congruence along the event horizon of a Black Hole:

$$-\frac{d\theta^H}{dt} + g_H \theta^H \simeq 8\pi \mathcal{I}^H, \quad (37)$$

$$-\frac{d\sigma_{ab}^H}{dt} + g_H \sigma_{ab}^H = \mathcal{E}_{ab}^H, \quad (38)$$

In the above expressions, $g_H = 2\pi T$, \mathcal{I}^H denotes the energy flux through the horizon and \mathcal{E}_{ab}^H is the Electric part of the Weyl tensor, i. e. the trace-free part of the Riemann tensor. For simplicity, the quadratic term containing σ in (37) has been neglected.

It is important to note that although Eqs. (37) and (38) look similar, actually they are quite different. This is because the source terms for (37) and (38) are \mathcal{I}^H and \mathcal{E}_{ab}^H respectively and they are different in nature.

While, \mathcal{I}^H has its origins from the matter-field, \mathcal{E}_{ab}^H — the electric part of the Weyl tensor corresponding to the Gravitational field — is related to the Gravitational waves⁴. For black-hole backgrounds, such ingoing gravitational waves are the Quasinormal modes. Quasinormal modes are

⁴ In case of vacuum solutions, the trace part of the Riemann tensor will vanish due to the Einstein equations, but the Weyl tensor can still be non-zero.

the modes of energy dissipation and they decay with time. The decay of these modes automatically allows only the causal response. Hence, the coefficient of Shear Viscosity is positive. However, the driving term \mathcal{I}^H for the volume expansion of the congruence cannot be damped as it is not driven by Quasinormal modes. Hence, imposing a teleological boundary condition leads to negative bulk viscosity. We note here that the ingoing boundary condition for the Gravitational waves at the Black Hole horizon is a physical one and is different from the boundary condition usually chosen in the Membrane paradigm. This difference has been noted in the literature in the context of the hydrodynamic limit of AdS-CFT[30].

V. DISCUSSION

It is known for a long time that the event horizon of a black-hole behaves like a viscous fluid. This suggests that Gravity is emergent. However, the Horizon fluid is unusual — it is a one parameter system and has negative bulk viscosity. There is no clear understanding of the statistical nature of such a system in the literature. In this work, we have tried to fill in that gap by constructing a theory of the fluctuations for this system. Such a theory goes further than fluid mechanics and provides a clear mathematical tool to evaluate certain parameters of the response functions whose values cannot be determined within fluid mechanics. It also establishes the statistical mechanical nature of the horizon-fluid on a firmer basis.

The analysis presented here is specific for Schwarzschild black-holes. However, the formalism developed here is sufficiently general to be applied to the fluid description for other black-holes. Also the fact that we use the Mean Field Theory to compute ζ does not make our analysis restrictive; it is model independent. Mean Field Theory fails to provide a good description of the horizon fluid when the Theory of Gravity is different from GR[32].

The calculation of the Coefficient of bulk viscosity, ζ of the horizon-fluid from the theory developed here leads to the exact value read off from the Navier-Stokes equation for the horizon-fluid. It also provides a natural explanation for the negative value of ζ by relating it with the teleological nature of black-hole event horizons. Any Statistical Mechanical explanation for why the Bulk Viscosity is negative has been lacking so far. This work addresses that gap in the literature and provides insight into this issue. An interesting corollary of our work is that the Coefficient of Bulk Viscosity for the fluid corresponding to a local horizon-like structure should be positive. One need not impose a future boundary condition in such a case [31] and the Coefficient of Bulk Viscosity is positive, thus providing further support for our claim. But the present work does not constitute a

full explanation. The shear equation for the null congruences on the horizon also has to be solved subject to a teleological boundary condition. Hence it has to be demonstrated that unlike ζ , the Coefficient of Shear Viscosity comes out to be positive in spite of this. Here we have provided a suggestion why η_S is positive. It is also significant that for a class of fluids, it has been shown in the literature[20], that an anti-H Theorem leads to negative transport coefficients. (It shows that entropy of the system decreases with time.) For horizon-fluid being the opposite of a normal fluid in this respect, it is expected that a H-Theorem would lead to negative ζ , whereas an anti-H Theorem would give rise to a positive value of ζ in this case.

Our analysis is the first step to take the fluid-gravity duality for asymptotically flat black-holes to the next level. For AdS-background black-holes, there already exists such a description via the CFT on the boundary[12],[13]. For general black-hole spacetimes, this is not the case. However, the present formalism can only address the transport properties of the horizon-fluid. It does not tell us whether a corresponding description exists on the Gravity side at the level of the fluctuations. This is a question that lies outside the scope of the present formalism.

Our formalism can be extended to compute the shear viscosity, electrical conductivity etc for the horizon-fluid. One can also look at other black-hole spacetimes like Kerr, Reisner-Nandstorm and so on. It would be interesting to have a Statistical Mechanical understanding as to why Black Holes in the AdS-background are different. However, as noted in [18], the horizon-fluid in AdS-background has a richer phase diagram, that shows the existence of a first order phase transition and a tri-critical point. So the Theory of the Fluctuations might have to be developed in a quite independent fashion for AdS-horizon-fluids. Such a task lies outside the scope of this paper. It would also be interesting to look at other theories of Gravity. Finally, it is well known that the Transport coefficients diverge as one reaches the critical point[17]. Our analysis here does not address this issue as we assume that we are sufficiently far from the critical point to neglect this. The phase transition that occurs in the Mean Field Theory of the horizon-fluid is such that only the ordered phase corresponds to the black-hole[18]. This means that as we start moving from the black-hole phase to some other phase, the Transport coefficients would change from the values given for a Horizon fluid and ultimately diverge as we come sufficiently close to the critical point. We hope to address this issue elsewhere.

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Appendix

Appendix A: Fluctuations of the Horizon Fluid

1. The Physical Process First Law for the horizon-fluid

The Physical process First Law for a black-hole Thermodynamics can be stated as follows: The mass of a black-hole increases as a result of mass-energy falling through the event horizon of the black-hole. The area of the cross section of the event horizon also increases in this process. Let us denote the temperature of the black-hole by T , the increase in mass by δM and the increase in the area of the horizon by δA . This law states that they are related by the following equation for a Schwarzschild black-hole [5, 6, 21],

$$\delta M = \frac{1}{4} T \delta A. \quad (\text{A1})$$

In the Fluid picture, this corresponds to the following process: Initially a fluid is in a state slightly away from equilibrium. It slowly moves towards the equilibrium state, which is the end state of this process. The change in the total energy of the fluid is related to the change in the volume(which is area because the horizon Fluid is $(2+1)$ -D) by (A1).

Using (11), it can be shown from this that[19], $\delta E = T \delta S$. This is the statement of the First Law of Thermodynamics for the fluid system.

2. Non-equilibrium dynamics and Transport process

For a system that is slightly away from equilibrium, Onsager's hypothesis[23] is applicable. Assuming k does not change in Eq. (8), we can describe the evolution of the order parameter η by the simplest form of the Langevin equation,

$$\ddot{\eta} = -\beta \dot{\eta} + F(t), \quad (\text{A2})$$

where, $F(t)$ is the random term and $\beta \dot{\eta}$ is the anti-damping term(Since the bulk viscosity of the horizon-fluid is negative.) due to the bulk viscosity of the fluid. Here we have taken only the effect

of bulk viscosity of the fluid as we take it to be homogeneous.

Now we define a quantity, $x = \frac{\dot{N}}{N} = \frac{\dot{A}}{A}$ and from (A2), it can be showed that[19],

$$\frac{d\langle x \rangle}{dt} = C T \langle x \rangle - \frac{1}{2} \langle x \rangle^2 + 8k\alpha\rho_d \quad (\text{A3})$$

where, ρ_d is the dissipated energy density and C is a constant.

Since, $x = \theta$, A , the area of the null congruence and t , the affine parameter along it[7]; (A3) is similar to the Raychaudhuri equation sans the shear term,

$$\frac{d\theta}{dt} = 2\pi T\theta - \frac{1}{2}\theta^2 - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta, \quad (\text{A4})$$

where, $C = -2\pi$ and

$$\rho_d = \frac{1}{8k\alpha}(8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta + \langle \dot{\eta} \rangle^2). \quad (\text{A5})$$

It is to be noted that except for the values of the coefficients, the signatures have turned out to be the same in both these equations.

Appendix B: The Computation of ζ in Frequency space

ζ can also be determined by computing the autocorrelation function for F_{Th} in the frequency space. The method we use here is again that given by Kubo in [16]. In this case, we deduce the formula for ζ from a generalised Langevin equation ab initio. As we shall see, this makes the underlying assumptions clear. We shall basically follow Kubo's approach[16] by adopting it to our case.

We start with a generalised Langevin equation which gives the dynamics of the transport process in which A changes by $\delta A(t)$. Following Kubo[16], we assume that the process $\delta A(t)$ is stationary at equilibrium. Also here we shall assume that the autocorrelation functions depend only on the time difference. As before, there is a thermodynamic force F_{Th} acting on the system. Then the Langevin equation can be written as

$$\delta \dot{A}(t) = - \int_{-\infty}^t \gamma(t-t') \delta A(t') dt' + F_{Th}(t). \quad (\text{B1})$$

Because of Hooke's law, $\bar{F}_{Th}(t) \propto \delta A(t)$. Then for $F_{Th}(t) = 0$, one has

$$\delta \dot{A}(t) = - \int_{-\infty}^t \gamma(t-t') \bar{F}_{Th}(t') dt'. \quad (\text{B2})$$

Taking Fourier transform on both sides of (B2), one gets,

$$\delta \dot{A}[\omega] = -\gamma[\omega] \bar{F}_{Th}[\omega]. \quad (\text{B3})$$

Since, $\delta\dot{A}[\omega] = -\delta A(t=0) + i\omega\delta A[\omega]$, we have

$$\delta A[\omega] = -\left(-\frac{i\gamma[\omega]}{\omega}\right)\bar{F}_{Th}[\omega]. \quad (\text{B4})$$

For bulk viscosity, one has the relation[23, 26],

$$\delta A[\omega] = -\zeta[\omega]\bar{F}_{Th}[\omega]A. \quad (\text{B5})$$

The factor A is present to signify that it is the area strain, $\frac{\delta A}{A}$, that is related to the stress \bar{F}_{Th} by the coefficient of bulk viscosity.

Comparing (B4) with (B5), one gets,

$$\zeta[\omega] = -\frac{i\gamma[\omega]}{A\omega}. \quad (\text{B6})$$

The above relation would be useful in determining ζ . Taking Fourier transform of (B1), we get,

$$\delta A[\omega] = \frac{1}{i\omega + \gamma[\omega]}F_{Th}[\omega]. \quad (\text{B7})$$

Let us now consider the Fourier transform of the autocorrelation function for $\delta\dot{A}(t)$. Using the condition of stationarity, we can write this as,

$$F.T.[\langle\delta\dot{A}(t_0)\delta\dot{A}(t_0+t)\rangle] = i\omega\langle\delta A^2(t_0)\rangle - (i\omega)^2 \int_0^\infty \exp -i\omega t \langle\delta A(t_0)\delta A(t_0+t)\rangle dt. \quad (\text{B8})$$

Now one can determine $F.T.[\langle\delta A(t_0)\delta A(t_0+t)\rangle]$ from (B1), when $F_{Th}(t) = 0$. In this case, (B1) gives after taking Fourier transform on both sides,

$$\int_0^\infty \exp -i\omega t \langle\delta A(0)\delta A(t)\rangle dt = \frac{\langle\delta A^2(0)\rangle}{i\omega + \gamma[\omega]}. \quad (\text{B9})$$

Using (B9) in (B8), one gets,

$$F.T.[\langle\delta\dot{A}(t_0)\delta\dot{A}(t_0+t)\rangle] = \frac{i\omega\gamma[\omega]}{i\omega + \gamma[\omega]}\langle\delta A^2(t_0)\rangle. \quad (\text{B10})$$

From the Langevin equation (B1), one gets,

$$\begin{aligned} F_{Th}(t_0) &= \delta\dot{A}(t_0), \\ F_{Th}(t_0+t) &= \delta\dot{A}(t_0+t) + \int_{t_0}^{t_0+t} \gamma(t_0+t-t')\delta A(t')dt'. \end{aligned} \quad (\text{B11})$$

From (B11), we can write down an expression for the autocorrelation function for F_{Th} as follows.

$$\langle F_{Th}(t_0)F_{Th}(t_0+t) \rangle = \langle\delta\dot{A}(t_0)\delta\dot{A}(t_0+t)\rangle + \int_{t_0}^{t_0+t} \gamma(t_0+t-t')\langle\delta\dot{A}(t_0)\delta A(t')\rangle dt'. \quad (\text{B12})$$

Taking the Fourier transform of (B12) on both sides and using (B8), we get,

$$F.T.[\langle F_{Th}(t_0)F_{Th}(t_0+t) \rangle] = \frac{i\omega\gamma[\omega]}{i\omega + \gamma[\omega]} \langle \delta A^2(t_0) \rangle + \gamma[\omega] \int_0^\infty \langle \delta \dot{A}(t_0)\delta A(t_0+t) \rangle \exp(-i\omega t) dt. \quad (B13)$$

After some algebra, one can write this as

$$F.T.[\langle F_{Th}(t_0)F_{Th}(t_0+t) \rangle] = \gamma[\omega] \langle \delta A^2(t_0) \rangle. \quad (B14)$$

Now we assume that the Principle of Equipartition holds and write (B14) as,

$$\gamma[\omega] = Im[\frac{1}{KT} \int_0^\infty \langle F_{Th}(t_0)F_{Th}(t_0+t) \rangle \exp(-i\omega t) dt]. \quad (B15)$$

Using (B6), one can write,

$$\zeta[\omega] = - \lim_{\omega \rightarrow \omega_S} Im[\frac{i}{AKT\omega} \int_0^\infty \langle F_{Th}(t_0)F_{Th}(t_0+t) \rangle \exp(-i\omega t) dt], \quad (B16)$$

where, ω_S is the frequency corresponding to the system size as discussed during the computation of ζ in Real space. At this point, we can use (25) to get,

$$\zeta[\omega] = - \lim_{\omega \rightarrow \omega_S} Im[\frac{i}{AKT\omega} P^2 \int_{-\infty}^\infty \langle \delta A(t)\delta A(0) \rangle \theta(-t) \exp(i(\bar{\omega} - \omega)t) dt]. \quad (B17)$$

(B17) follows from (B16) once we assume that $\delta A(t)$ has a time reversed solution such that $\delta A(t) = \delta A(-t)$. Now as in the computation of ζ in Real space, one has to assume that the fluctuating quantity obeys a damped wave equation and take the infrared limit. However, in this case, if we assume, $\delta A(t) = \delta A(0) \exp[\pm i(\omega + i\epsilon)t]$, then we would get a Dirac delta function. To avoid this, we assume that $\delta A(t)$ has a very small spread in the frequency. This allows us to write

$$\begin{aligned} \delta A(t) &= \delta A(0) \int_{\Delta\omega} \exp[i(\bar{\omega} + i\epsilon)t] d\bar{\omega}; \quad t > 0, \\ &= \delta A(0) \int_{\Delta\omega} \exp[-i(\bar{\omega} + i\epsilon)t] d\bar{\omega}; \quad t < 0, \end{aligned} \quad (B18)$$

where, $\Delta\omega$ is the spread in frequency of the fluctuating quantity. We can take it to be as small as we want. It is also to be noted that the condition, $\delta A(t) = \delta A(-t)$ is satisfied for this choice.

Using (B18) in (B17), we get,

$$\zeta[\omega] = - \frac{P^2 \langle \delta A^2(0) \rangle}{AKT\omega_S}. \quad (B19)$$

Proceeding in the same way as in the computation of ζ in Real Space, we have finally,

$$\zeta = - \frac{1}{16\pi}, \quad (B20)$$

as earlier.

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